UCRL-JC-134160
PREPRINT

Coupling Various Methods for Convection-Diffusion Problems with Applications to Flows in Porous Media

R.D. Lazarov
J.E. Pasciak
P.S. Vassilevski

This paper was prepared for submittal to the
Second Workshop on "Large-Scale Scientific Computations"
Sozopol, Bulgaria
June 2-6, 1999

May 6, 1999

Lawrence Livermore National Laboratory

ISC Texas A&M University CASC LLNL

Coupling Various Methods for Convection-Diffusion Problems with Applications to Flows in Porous Media

R.D. Lazarov*, J.E. Pasciak, and P.S. Vassilevski*

Large Scale Scientific Computations,

Sozopol, Bulgaria, June 2 - 6, 1999

*Work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract W-7405-Eng-48

Outline of the presentation

1. Introduction

2. Composite Formulations

2.1. Coupling Mixed & Galerkin Formulations

2.2. Coupling Mixed & Mixed Formulations

3. Discretization Methods

3.1. Coupling Mixed FE & Galerkin FE

3.2. Coupling Mixed FE & Finite Volumes

3.3. Coupling Mixed FE & Mixed FE

3.4. Coupling Finite Volume & Finite Volumes

4. Iterative Methods

4.1. Preconditioning of the Saddle-point System

4.2. Iterative Solution of the Reduced System

5. Numerical Experiments

1. Introduction

We consider the model convection–diffusion problem: given $f \in H^{-1}(\Omega)$, $a = a(x)$ s.p.d. matrix, find $p \in H^1(\Omega)$ s.t.

$$\begin{aligned} -\nabla \cdot a\nabla p + Cp &= f(x), && x \in \Omega, \\ \text{where} \quad Cp &= \nabla \cdot (p\underline{b}) + c_0 p, && x \in \Omega, \\ p(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Here Ω is a polygonal domain in $\mathcal{R}^d$, with $d = 2, 3$, the coefficients $c_0(x)$ and $\underline{b} = \underline{b}(x) = (b_1, \cdots, b_d)$ satisfy:

$$c_0(x) + \frac{1}{2}\nabla \cdot \underline{b}(x) \geq \gamma_0 = \text{const} > 0, \quad x \in \Omega.$$

If $\underline{b}(x) \equiv 0$ then we can have $\gamma_0 = 0$ This guarantees the coercivity of the problem and the existence of its solution in $H_0^1(\Omega)$.

Applications:

(1) heat and mass transfer;
(2) duffusion-reaction processes: $\underline{b} = 0$;
(3) flow in porous media: $\underline{b} = 0$, $c_0 = 0$;
(4) transport in porous media: $\underline{b}$ - given.

Example: **bioscreen model**.

Partition Ω into Ω_1 and Ω_2 with an interface Γ.

(1) In Ω_1 we can use:
(a) Mixed Method, or (b) Finite Volumes,

(2) In Ω_2 we can use one of:
(a) Galerkin FEM; (b) FV; (c) Mixed FEM.

2. Composite Formulations

We shall consider as a model situation when the doamain Ω is split into two subdomains with a common interface Γ, i.e. $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. In each subdomain we shall consider a separate formulation of the original problem.

Most of the methods extend to many subdomains. However, we have not studied the dependence of the parameters in the equivalence and a priori estimates on the **number of the subdomains**.

2.1. Coupling Mixed and Galerkin Methods

In Ω_1 we use mixed formulation, while in Ω_2 we shall use standard Galerkin weak fromulation.

In the mixed setting we introduce a new variable - velocity/flux: $\mathbf{u} = -a\nabla p$. To distinguish between the problem formulations we write

$$p_1 = p|_{\Omega_1} \text{ and } p_2 = p|_{\Omega_2}.$$

Different smoothness requirements on the components $\mathbf{u}$, p_1 and p_2 are imposed, namely

$$\begin{aligned} \mathbf{u} &\in H(div,\, \Omega_1), \\ p_1 &\in L^2(\Omega_1), \\ p_2 &\in H_0^1(\Omega_2,\, \partial\Omega_2 \setminus \Gamma) \equiv H_*^1(\Omega_2). \end{aligned}$$

Further, we shall us the standard notations for the Sobolev spaces for $i = 1, 2$:

$H^1_*(\Omega_i)$, $L^2(\Omega_i)$, $H(div, \Omega_1)$, $H^{1/2}_{00}(\Gamma)$, $H^{-1/2}(\Gamma)$.

In the inner product (or corresponding pairing) we shall skip the indexation by subdomain, i.e.

$$(v_i,\ w_i)_{\Omega_i} \equiv (v_i,\ w_i) = \int_{\Omega_i} v_i\, w_i\, dx.$$

And finally, by $< \cdot, \cdot >$ we shall denote the duality pairing between $H^{1/2}_{00}(\Gamma)$ and $H^{-1/2}(\Gamma)$ or simply the L^2-inner product on functions defined on Γ, i.e.

$$< f,\ g >= \int_\Gamma f\ g ds.$$

The weak composite formulation for the case of **pure diffusion** (i.e. $\underline{b}(x) \equiv 0$ and $c_0(x) \equiv 0$), which is a basis for the finite element method, is: find

$$
\begin{aligned}
&\mathbf{u} \in H(div, \Omega_1),\ p_1 \in L^2(\Omega_1),\ p_2 \in H^1_*(\Omega_2) \\
&(a^{-1}\mathbf{u},\ \mathbf{v}) - (p_1, \nabla \cdot \mathbf{v}) + < p_2,\ \mathbf{v} \cdot \mathbf{n}_1 > \qquad = 0, \\
&-(\nabla \cdot \mathbf{u},\ \phi) \qquad = (F, \phi), \\
&< \mathbf{u} \cdot \mathbf{n}_1,\ \psi > \qquad -(a \nabla p_2,\ \nabla \psi) \qquad = (F, \psi),
\end{aligned}
$$

for all

$$\mathbf{v} \in H(div, \Omega_1),\ \phi \in L_2(\Omega_1),\ \psi \in H^1_*(\Omega_2)$$

Remark 1. Note that this weak formulation does not involve Lagrange multipliers.

Theorem 1. The following *a priori* estimate is valid:

$$\|\mathbf{u}\|_{H(div,\,\Omega_1)} + \|p_1\|_{0,\,\Omega_1} + \|p_2\|_{1,\Omega_2} \leq C\left\{\|f\|_{0,\,\Omega_1} + \|f\|_{-1,\,\Omega_2}\right\}.$$

The proof usses the well–known inf–sup condition

$$\beta\|p_1\|_0 \leq \sup_{\mathbf{v}\in H(div,\,\Omega_1)} \frac{(p_1,\ \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{H(div,\,\Omega_1)}}.$$

The case of **convection-diffusion problems** is much more complicated.

To describe the weak form of the equation

$$\nabla \cdot \mathbf{u} + \nabla \cdot (\underline{b} p_1) + c_0 p_1 = f(x), \; x \in \Omega_1,$$

which is suitable for FE method, we shall need to allow for discontinuous functions p_1 from

$$H^1_{\text{loc}}(\Omega_1) = \left\{ \begin{array}{l} v_1 \in L^2(\Omega_1) : \exists \text{ partition } \mathcal{K} \\ \text{s.t. } v_1|_K \in H^1(K), \forall K \in \mathcal{K} \end{array} \right\}.$$

The functions in $H^1_{\text{loc}}(\Omega_1)$ have traces from both sides of the interfaces of the subdomains K. For $p_1 \in H^1_{\text{loc}}(\Omega_1)$ we denote these traces by p_1^o and p_1^i, where "o" stands for the outward (with respect to K) trace and "i" - the interior trace.

For any $K \in \mathcal{K}$ the advection-reaction operator Cp_1 contributes:

$$C_K(p_1,\, w_1) = \int_K (\nabla \cdot (\underline{b} p_1) + a_0(x) p_1)\, w_1\, dx \\ + \int_{\partial K_-} (p_1^o - p_1^i)\, w_1^i\, \underline{b} \cdot \mathbf{n}\, ds,$$

where $\Gamma_- = \{x \in \Gamma : \quad \underline{b}(x) \cdot \mathbf{n}_1 < 0\}$ is *inflow*, $\Gamma_+ = \{x \in \Gamma : \quad \underline{b}(x) \cdot \mathbf{n}_1 \geq 0\}$ is *outflow* boundary.

We integrate by parts in each K and sum for $K \in \mathcal{K}$. For $p_1, w_1 \in H^1_{\mathrm{loc}}(\Omega_1)$:

$$C(p_1, w_1) =$$

$$\sum \Big(- \int_K p_1\, \underline{b} \cdot \nabla w_1\, dx + \int_{\partial K_-} p_1^o\, w_1^i\, \underline{b} \cdot \mathbf{n}\, ds \\ + \int_K a_0(x)\, p_1\, w_1 dx + \int_{\partial K_+} p_1^i\, w_1^i\, \underline{b} \cdot \mathbf{n}\, ds \Big).$$

Note, that this from is defined for both continuous <u>and discontinuous with respect to $\mathcal{K}$ functions.</u>

If the subdomain K has a side/face on Γ_- then the trace p_1^o should be replaced it by its counterpart from Ω_2, namely by $p_2(x)$. Also on Γ_- we have $w_1^i = w_1$ and on $\partial\Omega_{1-} \setminus \Gamma_-$ we take $p_1^o = 0$. We get the following weak form of the second equation for all $w_1 \in H^1_{\text{loc}}(\Omega_1)$:

$$-(\nabla \cdot \mathbf{u}, w_1) - a_{11}(p_1, w_1) - a_{12}(p_2, w_1)$$

$$= -(f, w_1), \forall\ w_1 \in W_1.$$

Here

$$a_{11}(p_1,\ w_1) = -\sum_{K \in \mathcal{K}} \int_K p_1\ \underline{b} \cdot \nabla w_1\ dx + (c_0 p_1,\ w_1)$$
$$\sum_{K \in \mathcal{K}} \int_{\partial K \setminus \partial\Omega_{1-}} \left[(\underline{b} \cdot \mathbf{n}_1)_- p_1^o + (\underline{b} \cdot \mathbf{n}_1)_+ p_1^i\right] w_1^i ds$$
$$a_{12}(p_2,\ w_1) = \textstyle\int_{\Gamma_-} p_2(x)\ w_1(x)\ \underline{b} \cdot \mathbf{n}_1\ ds,$$

where for a given function $t(x)$ we have defined $t_- = \min(0, t)$ and $t_+ = \max(0, t)$.

Testing the original equation by a function $w_2 \in H_0^1(\Omega_2;\ \partial\Omega_2 \backslash \Gamma)$, using the zero boundary condition for w_2 on $\partial\Omega_2 \backslash \Gamma$, and the fact $\mathbf{u}\cdot\mathbf{n}_1 = -a\nabla p_1 \cdot \mathbf{n}_1 = a\nabla p_2 \cdot \mathbf{n}_2$ on Γ we get:

$$< \mathbf{u}\cdot\mathbf{n}_1,\ w_2 > -(a\nabla p_2,\ \nabla w_2)$$
$$-(\nabla\cdot(\underline{b}p_2), w_2) - (c_0 p_2, w_2) = -(f,\ w_2),$$

for all $w_2 \in H_0^1(\Omega_2;\ \partial\Omega_2 \setminus \Gamma)$. To account for the "inflow" boundary Γ_+ we need term $\int_{\Gamma_+} p_1 w_2\ \underline{b}\cdot\mathbf{n}_2\ ds$. We get it by adding it to the equation and subtracting its equal $-\int_{\Gamma_+} p_2 w_2\ \underline{b}\cdot\mathbf{n}_1\ ds$ since on Γ we have $p_1 = p_2$.

We get the following form of the last equation:

$$< \mathbf{u}\cdot\mathbf{n}_1,\ w_2 > -a_{21}(p_1, w_2) - a_{22}(p_2, w_2) = -(f, w_2),$$

for all $w_2 \in H_*^1(\Omega_2)$.

Summarizing, we have the following composite system: find

$$\mathbf{u} \in H(div,\ \Omega_1),\ p_1 \in H^1_{\text{loc}}(\Omega_1),\ p_2 \in H^1_*(\Omega_2)$$

$$\begin{aligned}
(a^{-1}\mathbf{u}, \mathbf{v}) \quad & - (p_1, \nabla \cdot \mathbf{v}) + < p_2, \mathbf{v} \cdot \mathbf{n}_1 > && = 0, \\
-(\nabla \cdot \mathbf{u}, w_1) \quad & - a_{11}(p_1, w_1) - a_{12}(p_2, w_1) && = -(f, w_1), \\
< \mathbf{u} \cdot \mathbf{n}_1, w_2 > & - a_{21}(p_1, w_2) - a_{22}(p_2, w_2) && = -(f,\ w_2),
\end{aligned}$$

for all

$$\mathbf{v} \in H(div,\ \Omega_1),\ w_1 \in H^1_{\text{loc}}(\Omega_1),\ w_2 \in H^1_*(\Omega_2)$$

where the bilinear forms are defined as:

$$\begin{aligned}
a_{11}(p_1,\ w_1) \quad & = \sum_{K \in \mathcal{K}} \int_{\partial K \setminus \partial \Omega_{1-}} \left[(\underline{b} \cdot \mathbf{n})_- p_1^o + (\underline{b} \cdot \mathbf{n})_+ p_1^i \right] w_1^i ds \\
& \quad - \sum_{K \in \mathcal{K}} \int_K p_1\ \underline{b} \cdot \nabla w_1\ dx + (c_0 p_1,\ w_1) \\
a_{12}(p_2,\ w_1) \quad & = \int_{\Gamma_-} p_2\ w_1\ \underline{b} \cdot \mathbf{n}_1\ ds, \\
a_{21}(p_1,\ w_2) \quad & = \int_{\Gamma_+} p_1\ w_2\ \underline{b} \cdot \mathbf{n}_2\ ds, \\
a_{22}(p_2,\ w_2) \quad & = (a \nabla p_2,\ \nabla w_2) + (\nabla \cdot (\underline{b} p_2),\ w_2) \\
& \qquad + (c_0 p_2, w_2) + \int_{\Gamma_-} p_2\ w_2\ \underline{b} \cdot \mathbf{n}_1\ ds.
\end{aligned}$$

The stability of this problem is in much weaker sense. Namely, we have:

Theorem 2. The solution of the composite problem exists and satisfies the following *a priori* estimate:

$$\|\mathbf{u}\|_{L^2(\Omega_1)} + \|p_1\|_{*,\,\Omega_1} + \|p_2\|_{*,\Omega_2} \le C\|f\|_{0,\,\Omega}$$

where

$$\begin{aligned}\|v_1\|^2_{*,\Omega_1} + \|v_2\|^2_{*,\Omega_2} = \\ & \frac{1}{2}\sum_{e\in\mathcal{E}_0}\int_e [v_1]^2|\underline{b}\cdot\mathbf{n}_1|\,ds + \gamma_0\|v_1\|^2_{0,\Omega_1} + \\ & \frac{1}{2}\int_\Gamma (v_1 - v_2)^2|\underline{b}\cdot\mathbf{n}_1|\,ds + \\ & (\tilde{a}\nabla v_2, \nabla v_2) + \gamma_0\|v_2\|^2_{0,\Omega_2}\end{aligned}$$

Take stable mixed pairs of spaces: $\mathbf{V}_i = H(div, \Omega_i)$, $W_i = L^2(\Omega_i)$, $i = 1, 2$, and $\Lambda_2 = H_0^{1/2}(\Gamma)$.

The combined mixed/mixed formulation is: find

$$
\boxed{
\begin{array}{ll}
(\mathbf{u}_1, p_1) \in \mathbf{V}_1 \times W_1,\ (\mathbf{u}_2, p_2) \in \mathbf{V}_2 \times W_2, & \lambda_2 \in \Lambda_2 \\
a_1(\mathbf{u}_1, \mathbf{v}_1) - (p_1, \nabla\cdot\mathbf{v}_1) + < \lambda_2, \mathbf{v}_1 \cdot \mathbf{n}_1 > & = 0,\ \mathbf{v}_1 \in \mathbf{V}_1, \\
-(\nabla\cdot\mathbf{u}_1,\ q_1) & = (f, q_1), q_1 \in W_1, \\
a_2(\mathbf{u}_2,\ \mathbf{v}_2) - (p_2,\ \nabla\cdot\mathbf{v}_2) + < \lambda_2, \mathbf{v}_2 \cdot \mathbf{n}_2 > & = 0,\ \mathbf{v}_2 \in \mathbf{V}_2, \\
-(\nabla\cdot\mathbf{u}_2,\ q_2) & = (f, q_2),\ q_2 \in W_2, \\
< \mathbf{u}_2 \cdot \mathbf{n}_2, \mu_2 > + < \mathbf{u}_1 \cdot \mathbf{n}_1, \mu_2 > & = 0,\ \mu_2 \in \Lambda_2.
\end{array}
}
$$

Here λ_2 plays a role of a Lagrange multiplier and

$$a_i(\mathbf{u}_i,\ \mathbf{v}_i) = \int_{\Omega_i} a^{-1}(x)\mathbf{u}_i \cdot \mathbf{v}_i dx,\ i = 1, 2.$$

2.3. Hybrid Formulation

This formulation is good for coupling two Galerkin FEMs, Galerkin FEM and FVM, and FVM with FVMs. We show it on two subdomains (the case of many subdomains has been studied by Ewing, L. Lin, and Lin).

First we denote by

$$H^1_*(\Omega_i) = \{v_i \in H^1(\Omega_i) : \quad v_i|_{\partial\Omega_i \setminus \Gamma} = 0\}.$$

The space X and its norm are defined as:

$$X = H^1_*(\Omega_1) \times H^1_*(\Omega_2), \quad \|v\|^2_X = \|v\|^2_{1,\Omega_1} + \|v\|^2_{1,\Omega_2}.$$

The space $H^1_0(\Omega)$ can be characterized by

$$H^1_0(\Omega) = \{v \in X : \quad < v_1 - v_2, \, \phi >_\Gamma = 0, \quad \forall \phi \in H^{-1/2}_{00}(\Gamma)\}.$$

Now we define the bilinear from $a^* : \; X \times X \to \mathcal{R}$:

$$a^*(v,w) = \sum_{i=1,2} \int_{\Omega_i} a(x) \nabla v_i \cdot \nabla w_i \, dx.$$

Thus the primal hybrid formulation is: find

$$(p_1,\ p_2,\ \psi) \in (H^1_*(\Omega_1),\ H^1_*(\Omega_2),\ H_{00}^{-\frac{1}{2}}(\Gamma))$$

such that

$$\boxed{\begin{array}{lll} a^*(p,q) + < q_1 - q_2, \psi > & = (f,p), & \forall q \in X, \\ < p_1 - p_2, \phi > & = 0, & \forall \phi \in H_{00}^{-\frac{1}{2}}(\Gamma). \end{array}}$$

One has the following result:

Theorem 3. The hybrid problem has unique solution $(p,\psi) \in X \times H_{00}^{-\frac{1}{2}}(\Gamma)$ and p is a solution to the original problem. Moreover, $\psi = a\nabla p_i \cdot \mathbf{n}_i,\ i = 1,2$ and

$$\|p\|_X + \|\psi\|_{-\frac{1}{2},\Gamma} \leq C\|f\|_{0,\Omega}.$$

3. Discretization Methods

We introduce **independent** triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of the domains Ω_1 and Ω_2 so they do not necessarily match along Γ and use different method in each of them. Why blending different methods ?

1. **Independent meshing** in the subdomains for practical reasons;
2. **Better or more appropriate** approximation by different methods;
3. **Necessity** of gluing together already existing implementations.

For example, **FVE method** has a large variety of up-winding and stabilizing techniques for convection-diffusion problems, while **mixed FEM** has local conservation properties and superconvergent approximation.

We take $(\mathbf{V},\ W_1)$ to be a stable pair of mixed finite element spaces: R-T, B-D-M, B-F-D-M, etc. Then the mixed FEM on Ω_1 will involve $\mathbf{u}_h \in \mathbf{V}$ and $p_{1,h} \in W_1$.

For the problem in Ω_2 we can apply:

(1) Standard Finite Element Approximation;
(Wieners and Wohlmuth'98, L.P.V.'99)

(2) Finite Volume Element Approximation;
(L., Pasciak, and Vassilevski'99)

(3) Mixed Finite Element Approximation.
(Arbogast, Cowsar, Yotov, Wheeler'96-98, and L. Pasciak, Vassievski'99).

The first two methods have similar formulations:

(1) The FE as Galerkin method with **solution and test** spaces of conforming linear elements;
(2) The FVE as Petrov-Galerkin method with a **solution** space of conforming linear FE and a **test** space of piecewise constants over a partition of the domain into finite volumes.

The mixed/mixed FE approximation requires Lagrange multipliers on the interface Γ.

All these approximations are stable in an appropriate norm and preconditioning based on Dirichlet-Neumann and Neumann-Dirichlet maps can be applied.

3.1. Coupling Mixed FE & Galerkin FEM

The **solution** space W_2 is the standard conforming space of piece-wise linear functions over $\mathcal{T}_2$. Then the approximation for the **pure diffusion** problem reads: find

$$\begin{aligned}
&\mathbf{u}_h \in \mathbf{V}, \quad p_{1,h} \in W_1, \text{ and } \quad p_{2,h} \in W_2 && \text{s.t. } : \\
&(a^{-1}\mathbf{u}_h, \underline{\chi}) - (p_{1,h}, \nabla \cdot \underline{\chi}) + < p_{2,h}, \underline{\chi} \cdot \mathbf{n}_1 > && = 0, \\
&-(\nabla \cdot \mathbf{u}_h,\ \phi) && = (F, \phi), \\
&< \mathbf{u}_h \cdot \mathbf{n}_1, \psi > \quad - \quad (a \nabla p_{2,h}, \nabla \psi) && = (F,\ \psi), \\
&\text{for all} \quad \underline{\chi} \in \mathbf{V},\ \phi \in W_1,\ \psi \in W_2.
\end{aligned}$$

This approximation was introduced by
– Wieners and Wohlmuth, 98,
– L., Pasciak, and Vassilevski, 99 proposed and studied several iterative methods based on:

(1) presentation of the system in block form as:

$$\mathcal{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{T}^T & \mathbf{N}^T \\ -\mathbf{T} & A_2 & 0 \\ \mathbf{N} & 0 & 0 \end{bmatrix}, \quad \mathcal{A}U = F, \quad U \in \mathcal{X}$$

where $\mathcal{X} = \mathbf{V}_h \times W_2 \times W_1$ and $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ and

(2) using Poincaré-Steklov operators (Neumann-Dirichlet and Dirichlet-Neumann maps for the interface data).

3.2. Mixed FE/FV Approximation

The finite volume approximation will be understood as Petrov-Galerkin method. For this we need also the **test** space denoted by W_2^* associated with the "triangulation" $\mathcal{T}_2^*$ of Ω_2 into finite (control) volumes.

.

Then W_2^* is the space of the piece-wise constant functions over the volumes from $\mathcal{T}_2^*$.

The ballance equation over a volume $V(x) \in \mathcal{T}_2^*$ is:

$$- \int_{\partial V} a \nabla v_2 \cdot \mathbf{n} ds = \int_V f dx, \quad v_2 \in W_2.$$

Introduce the form $a_{2,h}(v, \psi^*)$ for $v \in W_2$, $\psi^* \in W_2^*$:

$$a_{2,h}(v, \psi^*) = - \sum_x \psi^*(x) \int_{\partial V(x)} a \nabla v \cdot \mathbf{n} \, ds.$$

Then the **MFE/FV approximation** is: find

$\mathbf{u}_h \in \mathbf{V}$, $p_{1,h} \in W_1$, $p_{2,h} \in W_2$ s.t.

$$\begin{aligned}
&(a^{-1}\mathbf{u}_h, \mathbf{v}) - (p_{1,h}, \nabla \cdot \mathbf{v}) + < I_h^* p_{2,h}, \mathbf{v} \cdot \mathbf{n}_1 > = 0, \\
&-(\nabla \cdot \mathbf{u}_h, \ \phi) \qquad\qquad\qquad\qquad\qquad\qquad = (F, \phi), \\
&< \mathbf{u}_h \cdot \mathbf{n}_1, \psi^* > \qquad - \quad a_{2,h}(p_{2,h}, \psi^*) \ = (F, \psi^*),
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}$, $\phi \in W_1$, $\psi^* \in W_2^*$.

Here $I_h^* : \ W_2 \longmapsto W_h^*$ is a piece-wise constant interpolation operator.

Remark. To see in what sense the coupling provides continuity of the approximate presssure and the flux at the interface we need to define these quantities on Γ.

(1) A reasonable approximation of the flux/velocity over each FE in Ω_2 is $-a\nabla p_{2,h}$. Then $p_{2,h}$ satisfies:

$$< a\nabla p_{2,h} \cdot \mathbf{n}_1, \psi^* >_\Gamma + a_{2,h}(p_{2,h}, \psi^*) = (f, \psi^*)$$

for $\psi^* \in W_2^*$. Thus, the continuity of the flux across Γ is imposed in the following weak sense:

$$< (\mathbf{u}_h + a\nabla p_{2,h}) \cdot \mathbf{n}_1, \psi^* >_\Gamma = 0, \ \psi^* \in W_2^*.$$

(2) The continuity of the pressure is more complicated. Notice that in the mixed side the pressure is discontinuous across the FE interfaces. In general, the degrees of freedom are the medicenters of the FEs. In order to compute the pressure at a FE edge we need Taylor expansion by using the approximations of p and $\mathbf{u}$.

Another possibility is: find

$$\boxed{\begin{aligned}
&\mathbf{u}_h \in \mathbf{V},\ p_{1,h} \in W_1,\ p_{2,h} \in W_2 \ \text{ such that}\\
&(a^{-1}\mathbf{u}_h, \mathbf{v}) - (p_{1,h}, \nabla \cdot \mathbf{v}) + < p_{2,h}, \mathbf{v} \cdot \mathbf{n}_1 >= 0,\\
&-(\nabla \cdot \mathbf{u}_h,\ \phi) \qquad\qquad\qquad\qquad\qquad\qquad = (F, \phi),\\
&< \mathbf{u}_h \cdot \mathbf{n}_1, I_h \psi^* > \qquad\qquad -a_{2,h}(p_{2,h}, \psi^*) = (F, \psi^*),\\
&\forall\ \mathbf{v} \in \mathbf{V},\ \phi \in W_1,\ \psi^* \in W_2^*.
\end{aligned}}$$

Here $I_h : \ W_2^* \longmapsto W_h$ is a piece-wise linear interpolation operator. This is NOT locally conservative at the cells adjacent to Γ.

Theorem 4. The presented above approximations are stable and optimal with respect to the rate of convergence and regularity of the solution. Namely, the following error estimate is valid:

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(div,\Omega_1)} + \|p_1 - p_{1,h}\|_{0,\,\Omega_1} + \\ + \|p_2 - p_{2,h}\|_{1,h,\Omega_2} \leq \\ C\left\{h_1\|\mathbf{u}\|_{1,\Omega_1} + h_1\|p_2\|_{1,\Omega_1} + h_2\|p_2\|_{2,\Omega_2}\right\}$$

with a constant C independent of h.

3.3. Mixed & Mixed Discretizations

We take stable mixed pairs of spaces $(\mathbf{V}_1, W_1)$ and $(\mathbf{V}_2, W_2)$ in Ω_1 and Ω_2, respectively. Next, we take $\Lambda_2 \subset H_0^1(\Gamma)$ a conforming FE space.

Then the combined mixed/mixed FEM is: find

$$\boxed{\begin{array}{ll}
(\mathbf{u}_1, p_1) \in \mathbf{V}_1 \times W_1, \ (\mathbf{u}_2, p_2) \in \mathbf{V}_2 \times W_2, & \lambda_2 \in \Lambda_2 \\
a_1(\mathbf{u}_1, \mathbf{v}_1) - (p_1, \nabla\cdot\mathbf{v}_1) + < \lambda_2, \mathbf{v}_1 \cdot \mathbf{n}_1 > & = 0, \ \mathbf{v}_1 \in \mathbf{V}_1, \\
-(\nabla\cdot\mathbf{u}_1, \ q_1) & = (F, q_1), \ q_1 \in W_1, \\
a_2(\mathbf{u}_2, \ \mathbf{v}_2) - (p_2, \ \nabla\cdot\mathbf{v}_2) + < \lambda_2, \mathbf{v}_2 \cdot \mathbf{n}_2 > & = 0, \ \mathbf{v}_2 \in \mathbf{V}_2, \\
-(\nabla\cdot\mathbf{u}_2, \ q_2) & = (F, q_2), \ q_2 \in W_2, \\
< \mathbf{u}_2 \cdot \mathbf{n}_2, \mu_2 > + < \mathbf{u}_1 \cdot \mathbf{n}_1, \mu_2 > & = 0, \ \mu_2 \in \Lambda_2.
\end{array}}$$

Here λ_2 plays a role of a Lagrange multiplier and

$$a_i(\mathbf{u}_i, \ \mathbf{v}_i) = \int_{\Omega_i} a^{-1}(x)\mathbf{u}_i \cdot \mathbf{v}_i dx, \ i = 1, 2.$$

3.4. Coupling FV/FV Approximation

We need to formulate the finite element (solution) and finte volume (test) spaces for this setting. Since each doamin has its own triangulation we introduce the finite element approximation X_h of $X = H^1_*(\Omega_1) \times H^1_*(\Omega_2)$:

$$X_h = \{v_i \in C(\Omega_i) \cap H^1_*(\Omega_i) : \; v_i|_T \in P_1(T), \; \forall T \in \mathcal{T}_i\}.$$

In fact, we have that $X_h \equiv W_1 \times W_2$, where W_i is the conforming finite element spaces in Ω_i, $i = 1, 2$.

Next, we define the FE space for the Lagrange multipliers. We shall choose it in the simplst poosible way, as the space of the traces of the functions from W_1 modified at the end points, i.e.

$$M_h = \left\{ \begin{array}{r} \phi \in C(\Gamma) : \; \phi \text{ is linear on } \; \mathcal{T}_1 \text{ and is a} \\ \text{constant at end intervals} \end{array} \right\}.$$

We introduce

$$a^{FV}(p, q^*) = a_{1,h}(p_1, q_1^*) + a_{2,h}(p_2, q_2^*), \forall p \in V_h, q^* \in X_h^*.$$

The **mortar finte volume element** method is: find $p_h = (p_{1,h}, p_{2,h}) \in X_h$ and $\psi_h \in M_h$ s.t.

(1) locally non-conservative along Γ:

$$\begin{aligned} a^{FV}(p_h, I_h^* q) + < q_1 - q_2, \psi_h > &= (f, I_h^* q), \ \forall q \in X_h \\ < p_{1,h} - p_{2,h}, \phi > &= 0, \ \forall \phi \in M_h. \end{aligned}$$

(2) locally conservative scheme:

$$\begin{aligned} a^{FV}(p_h, I_h^* q) + < I_h^*(q_1 - q_2), \psi_h > \ \forall q \in X_h \\ < I_h^*(p_{1,h} - p_{2,h}), \phi > &= 0, \ \forall \phi \in \end{aligned}$$

The following result is a particular case of the many subdomains ($i > 2$) of Ewing, L., Lin, and Lin, 99:

Theorem 5. If $p_i \in H^{1+\tau_i}(\Omega_i)$, with $0 < \tau_i \leq 1$, $i = 1, 2$ then the solution of the mortar FVM satisfies the error estimate:

$$\begin{aligned} \|p - p_h\|_X & + \|\psi - \psi_h\|_{-\frac{1}{2},\Gamma} \\ & \leq C \sum_{i=1,2} (h_i^{\tau_i} \|p_i\|_{1,\Omega_i} + h_i \|f\|_{0,\Omega_i}) \end{aligned}$$

with a constant C independent of h.

Note that this theorem quarantees the convergence of the Lagrange multipiers (which have meaning of the fluxes accross Γ) in the negative 1/2-norm.

4. Iterative solution of the composite saddle–point problem

We consider coupling of mixed and Galerkin methods. Other coupled discretizations use the same construction. Let $\mathcal{X}$ denote the product space $\mathbf{V}_h \times W_2 \times W_1$ and consider the operator $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{T}^T & \mathbf{N}^T \\ -\mathbf{T} & A_2 & 0 \\ \mathbf{N} & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned}
(\mathbf{A}_1\underline{\chi}, \underline{\theta}) &= (a^{-1}\underline{\chi}, \underline{\theta}) && \text{for all } \underline{\chi}, \underline{\theta} \in \mathbf{V}, \\
(\mathbf{N}\underline{\chi}, w_1) = (\mathbf{N}^T w_1, \underline{\chi}) &= -(\nabla \cdot \underline{\chi}, w_1) && \\
& && \text{for all } \underline{\chi} \in \mathbf{V},\ w_1 \in W_1, \\
(\mathbf{T}\underline{\chi}, w_2) = (\mathbf{T}^T w_2, \underline{\chi}) &= \langle w_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma && \\
& && \text{for all } \underline{\chi} \in \mathbf{V},\ w_2 \in W_2, \\
(A_2 v_2, w_2) = a(v_2, w_2) &\equiv (a\nabla v_2, \nabla w_2) && \\
& && \text{for all } v_2, w_2 \in W_2.
\end{aligned}$$

4.1. Direct preconditioning of the saddle-point system

We take the block diagonal operator

$$\mathcal{D} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

where $(\Lambda \underline{\chi},\ \underline{\theta}) = (a^{-1}\underline{\chi},\ \underline{\theta}) + (\nabla \cdot \underline{\chi},\ \nabla \cdot \underline{\theta})$ for all $\underline{\chi},\ \underline{\theta} \in \mathbf{V}$.

By the stability of the saddle–point system,

$$c_0|||\mathbf{U}|||_{\mathcal{D}}^2 \le |||\mathcal{A}\mathbf{U}|||_{\mathcal{D}}^2 = \sup_{\theta \in \mathcal{X}} \frac{(\mathcal{A}\mathbf{U}, \theta)^2}{(\mathcal{D}\theta, \theta)} \le c_1|||\mathbf{U}|||_{\mathcal{D}}^2,$$

for any $\mathbf{U} \in \mathcal{X}$. Here $|||\cdot|||_{\mathcal{D}} = (\mathcal{D}\cdot, \cdot)^{1/2}$ and $(\cdot, \cdot)$ denotes the inner-product in the product space $\mathcal{X}$.

4.2. Iterative solution of the reduced Mixed FE/FVM problem

This applies to all combinations: mixed – standard Galerkin, mixed – finite volume, and mixed – mixed. The technique is based on Dirichlet–Neumann and Neumann–Dirichlet maps.

1. Elimination Procedure: solve two independent problems;
(a) the mixed problem in Ω_1:

$$\begin{aligned} (a^{-1}\mathbf{u}_h^0,\ \mathbf{v}) \quad -(p_{1,h}^0, \nabla\cdot\mathbf{v}) &= 0, \quad \mathbf{v}\in\mathbf{V},\\ -(\nabla\cdot\mathbf{u}_h^0,\ q_1) \qquad\qquad &= -(f,q_1), \quad q_1\in W_1. \end{aligned}$$

On Γ we have **homogeneous Dirichlet** data for $p_{1,h}^0$.

(b) the standard FVM in Ω_2:

$$a_{2,h}(p_{2,h}^0,\ q_2) = (f,q_2), \quad q_2\in W_2^*.$$

Here on Γ we have **homogeneous Neumann** data for $p_{2,h}^0$.

2. Iteration Procedure: the differences $\widehat{\mathbf{u}}_h = \mathbf{u}_h - \mathbf{u}_h^0$ and $\widehat{p}_{i,h} = p_i - p_{i,h}^0$ satisfy for $\mathbf{v}\in\mathbf{V}$, $q_1\in W_1$ and $q_2^*\in W_2^*$:

$$\begin{aligned} (a^{-1}\widehat{\mathbf{u}}_h,\mathbf{v}) - (\widehat{p}_{1,h},\nabla\cdot\mathbf{v}) + < I_h^*\widehat{p}_{2,h}, \mathbf{v}\cdot\mathbf{n}_1 > &= -< I_h^* p_{2,h}^0, \mathbf{v}\cdot\mathbf{n}_1 >,\\ (\nabla\cdot\widehat{\mathbf{u}}_h, q_1) &= 0.\\ < \widehat{\mathbf{u}}_h\cdot\mathbf{n}_1,\ q_2^* > + a_{2,h}(\widehat{p}_{2,h}, q_2^*) &= -< \mathbf{u}_h^0\cdot\mathbf{n}_1,\ q_2^* > . \end{aligned} \tag{1}$$

Define the trace space of FE space W_2 on Γ: $\Lambda_2 = W_2|_\Gamma$.

We introduce the maps (called often Poincaré-Steklov):

1. the **Dirichlet–Neumann** map $E_1 : \Lambda_2 \mapsto \mathbf{V} \cdot \mathbf{n}_1|_\Gamma$, defined for any $\lambda_2 \in \Lambda_2$ as the normal trace $\mathbf{w}_h[\lambda_2] \cdot \mathbf{n}_1$ of the mixed FE solution $\mathbf{w}_h[\lambda_2]$, i.e. $E_1\lambda_2 = \mathbf{w}_h[\lambda_2] \cdot \mathbf{n}_1|_\Gamma$, where

$$\begin{aligned}(a^{-1}\mathbf{w}_h[\lambda_2],\ \mathbf{v}) - (p_{1,h}[\lambda_2],\ \nabla\cdot\mathbf{v}) &= -< I_h^*\lambda_2, \mathbf{v}\cdot\mathbf{n}_1 >, \qquad && \mathbf{v} \in \mathbf{V},\\ (\nabla\cdot\mathbf{w}_h[\lambda_2],\ q_1) &= 0, && q_1 \in W_1.\end{aligned}$$

2. the **Neumann–Dirichlet** map $S_2 : L^2(\Gamma) \mapsto \Lambda_2$ defined for any $\lambda^* \in L^2(\Gamma)$ as $S_2\lambda^* \equiv \lambda_2 = p_{2,h}[\lambda^*]|_\Gamma$ obtained by the FVE method:

$$a_{2,h}(p_{2,h}[\lambda^*],\ I_h^* q_2) =< \lambda^*, I_h^* q_2 >, \qquad q_2 \in W_2.$$

In terms of the operators E_1 and S_2 one can rewrite the coupled system (1) in the form: if $\lambda^* = -\mathbf{u}_h^0 \cdot \mathbf{u}$, then

$$\begin{aligned}\widehat{\mathbf{u}}_h \cdot \mathbf{n}_1 &= E_1\lambda_2,\\ \lambda_2 &= S_2(\lambda^* - \widehat{\mathbf{u}}_h \cdot \mathbf{n}_1) = S_2(\lambda^* - E_1\lambda_2).\end{aligned}$$

Then, the reduced system reads:

$$(I + E_1 S_2)\widehat{\mathbf{u}}_h \cdot \mathbf{n}_1 = E_1 S_2 \lambda^*. \tag{2}$$

Now we define the extension $\tilde{\mathbf{w}}_1[\lambda_1^*]$ for any $\lambda_1^* \in \Lambda_1^* = \mathbf{V}\cdot\mathbf{n}_1|_\Gamma$, obtained by solving the problem

$$\boxed{\begin{array}{llll} (a^{-1}\tilde{\mathbf{w}}_h[\lambda_1^*],\ \mathbf{v}) & -(p_{1,h}[\lambda_1^*],\ \nabla\cdot\mathbf{v}) & = 0, & \mathbf{v}\in\mathbf{V},\ \mathbf{v}\cdot\mathbf{n}_1 = 0,\\ -(\nabla\cdot\tilde{\mathbf{w}}_h[\lambda_1^*],\ q_1) & & = 0, & q_1 \in W_1\\ & \tilde{\mathbf{w}}_h[\lambda_1^*]\cdot\mathbf{n}_1 & = \lambda_1^*. & \end{array}}$$

Consider the $H^{-1/2}$–equivalent on Λ_1^* inner product,

$$< \mu_1^*,\ \lambda_1^* >_* = (\tilde{\mathbf{w}}_h[\lambda_1^*],\ \tilde{\mathbf{w}}_h[\mu_1^*]). \qquad (3)$$

Theorem 6. The reduced problem (2) is well conditioned in this inner product. Namely, the following estimates hold:

$$< E_1 S_2 \lambda_1^*, \lambda_1^* >_* \ = a_{2,h}(p_{2,h}[\lambda_1^*],\ I_h^* p_{2,h}[\lambda_1^*]) \geq 0$$

and

$$\begin{aligned} < E_1 S_2 \lambda_1^*, \mu_1^* >_* \ &= a_{2,h}(p_{2,h}[\lambda_1^*],\ I_h^* p_{2,h}[\mu_1^*]) \\ &\leq C\|p_{2,h}[\lambda_1^*]\|_{1,\Omega_1}\|p_{2,h}[\mu_1^*]\|_{1,\Omega_1} \\ &\leq C\|\lambda_1^*\|_{-1/2,\Gamma}\cdot\|\mu_1^*\|_{-1/2,\Gamma}. \end{aligned}$$

These estimates guarantee that the preconditioned MINRES in the inner product $< \cdot,\cdot >_*$ will have optimal convergence rates.

4.3. Iterative solution of the reduced Mixed/Mixed FE problem

1. Elimination Procedure: solve two independent mixed problems;

$$
\begin{aligned}
a_1(\mathbf{u}_1^0,\ \mathbf{v}_1) \quad -(p_1^0,\ \nabla\cdot\mathbf{v}_1) &= 0, \quad \mathbf{v}_1 \in \mathbf{V}_1,\\
-(\nabla\cdot\mathbf{u}_1^0,\ q_1) &= -(f_1, q_1), \quad q_1 \in W_1.
\end{aligned}
$$

$$
\begin{aligned}
a_2(\mathbf{u}_2^0,\ \mathbf{v}_2) \quad -(p_2^0,\ \nabla\cdot\mathbf{v}_2) &= 0, \quad \mathbf{v}_2 \in \mathbf{V}_2,\\
-(\nabla\cdot\mathbf{u}_2^0,\ q_2) &= -(f_2, q_2), \quad q_2 \in W_2.
\end{aligned}
$$

2. Iteration Procedure: the difference $\widehat{\mathbf{u}}_i = \mathbf{u}_i - \mathbf{u}_i^0$ and $\widehat{p}_i = p_i - p_i^0$ and λ_2 satisfy the homogeneous problems,

$$
\begin{aligned}
a_1(\widehat{\mathbf{u}}_1,\ \mathbf{v}_1) - (\widehat{p}_1,\ \nabla\cdot\mathbf{v}_1) + < \lambda_2,\ \mathbf{v}_1\cdot\mathbf{n}_1 > &= 0, \quad \mathbf{v}_1 \in \mathbf{V}_1,\\
-(\nabla\cdot\widehat{\mathbf{u}}_1,\ q_1) &= 0, \quad q_1 \in W_1.
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
a_2(\widehat{\mathbf{u}}_2,\ \mathbf{v}_2) - (\widehat{p}_2,\ \nabla\cdot\mathbf{v}_2) + < \lambda_2,\ \mathbf{v}_2\cdot\mathbf{n}_2 > &= 0, \quad \mathbf{v}_2 \in \mathbf{V}_2,\\
-(\nabla\cdot\widehat{\mathbf{u}}_2,\ q_2) &= 0, \quad q_2 \in W_2,\\
< \widehat{\mathbf{u}}_2\cdot\mathbf{n}_2,\ \mu_2 > + < \widehat{\mathbf{u}}_1\cdot\mathbf{n}_1,\ \mu_2 > &= < \lambda^*,\ \mu_2 >, \ \mu_2 \in \Lambda_2.
\end{aligned}
\tag{5}
$$

Here, $\lambda^* = -\mathbf{u}_2^0\cdot\mathbf{n}_2 - \mathbf{u}_1^0\cdot\mathbf{n}_1$.

Introduce:

1. **Dirichlet–Neumann** map $E_1 : \Lambda_2 \mapsto \mathbf{V}_1 \cdot \mathbf{n}_1|_\Gamma$,

$$\boxed{E_1\lambda_2 = \mathbf{w}_1[\lambda_2] \cdot \mathbf{n}_1|_\Gamma}$$

defined for any $\lambda_2 \in \Lambda_2$ as the normal trace of the solution $\mathbf{w}_1[\lambda_2]$ of the mixed problem

$$\begin{aligned} a_1(\mathbf{w}_1[\lambda_2], \mathbf{v}_1) \quad -(p_1[\lambda_2], \nabla\cdot\mathbf{v}_1) \quad &= - < \lambda_2, \mathbf{v}_1 \cdot \mathbf{n}_1 >, \\ &\qquad \mathbf{v}_1 \in \mathbf{V}_1, \\ -(\nabla\cdot\mathbf{w}_1[\lambda_2], q_1) \quad &= 0, \quad q_1 \in W_1 : \end{aligned}$$

2. **Neumann–Dirichlet** map $S_2 : \; H^{-\frac{1}{2}}(\Gamma) \mapsto \Lambda_2$ defined for any $\lambda^* \in H^{-\frac{1}{2}}(\Gamma)$

$$\boxed{S_2\lambda^* = \lambda_2[\lambda^*].}$$

as the solution $\lambda_2[\lambda^*]$ of the hybrid mixed problem,

$$\begin{aligned} a_2(\mathbf{w}_2, \mathbf{v}_2) - (p_2, \nabla\cdot\mathbf{v}_2) + < \lambda_2[\lambda^*], \mathbf{v}_2 \cdot \mathbf{n}_2 > \quad &= 0, \quad \mathbf{v}_2 \in \mathbf{V}_2, \\ -(\nabla\cdot\mathbf{w}_2, q_2) \quad &= 0, \quad q_2 \in W_2, \\ < \mathbf{w}_2 \cdot \mathbf{n}_2, \mu_2 > \quad &= < \lambda^*, \mu_2 >, \\ &\qquad \mu_2 \in \Lambda_2. \end{aligned}$$

In terms of the operators E_1 and S_2 one can rewrite the coupled system (4)-(5) as follows:

$$\begin{aligned} \widehat{\mathbf{u}}_1 \cdot \mathbf{n}_1 &= E_1\lambda_2, \\ \lambda_2 &= S_2(\lambda^* - \widehat{\mathbf{u}}_1 \cdot \mathbf{n}_1) = S_2(\lambda^* - E_1\lambda_2). \end{aligned}$$

That is, the reduced system reads:

$$(I + S_2E_1)\lambda_2 = S_2\lambda^*. \tag{6}$$

Another reduction is also possible; namely, one has:

$$(I + E_1S_2)\widehat{\mathbf{u}}_1 \cdot \mathbf{n}_1 = E_1S_2\lambda^*. \tag{7}$$

Consider now the extension $\tilde{\mathbf{w}}_1[\lambda_1^*]$ for any $\lambda_1^* \in \Lambda_1^* = \mathbf{V}_1 \cdot \mathbf{n}_1|_\Gamma$, obtained by solving the problem

$$\boxed{\begin{array}{lrl} a_1(\tilde{\mathbf{w}}_1[\lambda_1^*],\, \mathbf{v}_1) & -(p_1[\lambda_1^*],\, \nabla\cdot\mathbf{v}_1) & = 0, \quad \mathbf{v}_1 \in \mathbf{V}_1, \\ & & \mathbf{v}_1 \cdot \mathbf{n}_1 = 0, \\ -(\nabla\cdot\tilde{\mathbf{w}}_1[\lambda_1^*],\, q_1) & & = 0, \quad q_1 \in W_1 \\ & \tilde{\mathbf{w}}_1[\lambda_1^*] \cdot \mathbf{n}_1 & = \lambda_1^*. \end{array}}$$

Consider the $H^{-1/2}$–equivalent inner product,

$$< \mu_1^*,\, \lambda_1^* >_* = a_1(\tilde{\mathbf{w}}_1[\lambda_1^*],\, \tilde{\mathbf{w}}_1[\mu_1^*]).$$

Theorem 7. Then the reduced problem (7) is symmetric and positive definite in the $< \cdot,\cdot >_*$– inner product. And the following spectral equivalence estimates hold:

$$\begin{aligned} < \mu_1^*, \mu_1^* >_* \le & < \mu_1^*, \mu_1^* >_* + < E_1 S_2 \mu_1^*, \mu_1^* >_* \\ \le & \ (1 + \text{const}) < \mu_1^*, \mu_1^* >_*, \ \mu_1^* \in \Lambda_1^*. \end{aligned}$$

Next we define the inner product,

$$< \lambda_2,\ \mu_2 >_{**} \equiv a_2(\tilde{\mathbf{w}}_2[\lambda_2],\ \tilde{\mathbf{w}}_2[\mu_2]),$$

where for a given $\theta_2 \in \Lambda_2$, $\tilde{\mathbf{w}}_2[\theta_2]$ is the "mixed harmonic extension" of θ_2, i.e. it is the solution of the mixed problem,

$$\boxed{\begin{aligned} a_2(\tilde{\mathbf{w}}_2[\theta_2],\ \mathbf{v}_2) - (p_2[\theta_2],\ \nabla\cdot\mathbf{v}_2) &= -< \theta_2,\ \mathbf{v}_2\cdot\mathbf{n}_2 >, \\ &\qquad \mathbf{v}_2 \in \mathbf{V}_2, \\ (\nabla\cdot\tilde{\mathbf{w}}_2[\theta_2],\ q_2) &= 0, \quad q_2 \in W_2. \end{aligned}}$$

Theorem 8. The reduced problem (6) is well–conditioned in this inner product. More specifically, the operator $I + S_2E_1$ is a symmetric, uniformly positive definite and uniformly bounded operator in the $< .,. >_{**}$–inner product.

5. Numerical Experiments

Test example:

- the domain is $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = (0,1) \times (0,1)$, $\Gamma = \{(1,y),\ 0 < y < b\}$, $b < 1$ is a given parameter, and $\Omega_2 = (1, 1+b) \times (0,b)$;
- the elliptic problem in Ω_1 is $-\nabla \cdot a_1 \nabla p_1 = f_1$, where the coefficient matrix
$$a_1 = \begin{bmatrix} 1 + 10x^2 + y^2 & \frac{1}{2} + x^2 + y^2 \\ \frac{1}{2} + x^2 + y^2 & 1 + x^2 + 10y^2 \end{bmatrix};$$
the exact solution is $p_1(x,y) = (1-x)^2 x (1-y) y$, hence $\mathbf{u} = -a_1 \nabla p_1$.
- the elliptic problem in Ω_2 is $-\Delta p_2 = f_2$, where the coefficient matrix is just the identity, i.e. $a_2 = I$, and the exact solution is $p_2(x,y) = 10^5 (1 + b - x)(x-1)^2 y (b - y)$.

5.1. Mixed & standard Galerkin tests

We used the following solution methods:

1. the MINRES (minimum residual method) with with the block-diagonal preconditioner

$$\begin{bmatrix} \mathbf{B}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_2 \end{bmatrix};$$

2. CG method applied to the reduced problem

$$(I + S_2 E_1) q_{2,h} = S_2(\mathbf{u}_h^0 \cdot \mathbf{n}_1 - E_1 p_{2,h}^0).$$

The stopping criterion: until the relative residual is reduced by a factor of 10^{-6}.

Error and iteration counts for $b = 0.55$.

	$h_1 = 1/16$ $h_2 = b/16$	$h_1 = 1/32$ $h_2 = b/32$	$h_1 = 1/64$ $h_2 = b/64$	$h_1 = 1/128$ $h_2 = b/128$	$\approx$ order
δ_p	3.18e-2	7.57e-3	1.83e-3	4.57e-4	2
δ_{u_1}	0.5749	0.1343	3.27e-2	7.87e-3	2
δ_{u_2}	0.3617	8.87e-2	2.21e-2	5.51e-3	2
$\delta_{u_{\text{int}}}$	0.3792	9.42e-2	2.37e-2	5.93e-3	2
δ_{p_2}	0.1519	3.44e-2	7.71e-3	1.91e-3	2
#	57	71	86	92	
ϱ	0.69	0.74	0.78	0.79	

Number of CG iterations and average reduction factors for solving the system $(I + S_2 E_1) q_{2,h} = rhs_{2,h}$; $b = 0.55$.

h_1	h_2: $b/16$	$b/32$	$b/64$	$b/128$
1/16	11, 0.21	12, 0.26	13, 0.30	13, 0.30
1/32	12, 0.30	15, 0.39	15, 0.39	15, 0.39
1/64	10, 0.22	14, 0.36	16, 0.39	15, 0.39
1/128	9, 0.21	11, 0.27	15, 0.38	16, 0.40

5.2. Mixed & mixed discretization tests

The same test problem, now discretized by a hybrid mixed method. As described above one reduces the coupled system to (7), i.e.,

$$(I + E_1 S_2)\hat{\mathbf{u}}_{1,h} \cdot \mathbf{n}_1 = E_1 S_2 \lambda^*,$$

where $\lambda^* = -\mathbf{u}^0_{2,h} \cdot \mathbf{n}_2 - \mathbf{u}^0_{1,h} \cdot \mathbf{n}_1$.

We write the resulting linear system in terms of the normal trace of the discrete solution, i.e., $\mathbf{u}_{1,h} \cdot \mathbf{n}_1$,

$$(I + E_1 S_2)\mathbf{u}_{1,h} \cdot \mathbf{n}_1 = \mathbf{rhs}_{1,h} \qquad (8)$$

We used the CG method applied to (8) in the inner product $< \cdot, \cdot >_*$ so that $I + E_1 S_2$ is symmetric and positive definite in that inner product. The stopping criterion: until relative residual is reduced by a factor 10^{-6}.

Error and iteration counts for problem (8); $b = 0.55$.

	$h_1 = 1/16$ $h_2 = b/16$	$h_1 = 1/32$ $h_2 = b/32$	$h_1 = 1/64$ $h_2 = b/64$	$h_1 = 1/128$ $h_2 = b/64$	$\approx$ order
$\delta_{-1/2}$	0.1601	3.97e-2	9.89e-3	2.47e-3	2
δ_0	0.9016	0.2341	6.38e-2	1.59e-2	2
#	6	7	8	8	
ϱ	0.08	0.11	0.15	0.17	

The second test is for the reduced problem (8) with a random $\mathbf{rhs}^*_{1,\,h}$.

Number of CG iterations and average reduction factors for solving the system $(I + E_1 S_2)\lambda^*_{1,h} = \mathbf{rhs}^*_{1,h}$; $b = 0.55$.

h_1	h_2: $b/16$	$b/32$	$b/64$	$b/128$
1/16	6, 0.08	7, 0.09	7, 0.13	7, 0.13
1/32	7, 0.10	7, 0.13	8, 0.16	8, 0.17
1/64	10, 0.22	9, 0.19	8, 0.17	9, 0.18
1/128	11, 0.21	11, 0.26	9, 0.20	8, 0.16